Conformally Invariant Random Geometry on Manifolds of Even Dimensions

Eva Kopfer

Hausdorff Center for Mathematics
Institute for Applied Mathematics

Universität Bonn

based on joint work
Conformally invariant random fields, Liouville Quantum Gravity measures, and random Paneitz operators on Riemannian manifolds of even dimension
with Lorenzo Dello Schiavo, Ronan Herry and Theo Sturm

Random Riemannian Geometry and Conformal Invariance

- (M, g) compact Riemannian manifold of even dimension n
- Conformal class

$$(M, [g]) := \{(M, g') : g' = e^{2\varphi} g, \varphi \in \mathcal{C}^\infty(M)\}$$

Naive Goal

Define probability measures $\mathbf{P}_{M,g}$ on $\mathcal{C}^\infty(M)$ such that

- $\mathbf{P}_{M,g'} = \mathbf{P}_{M,g}$ if $g' = e^{2\varphi} g$ for $\varphi \in \mathcal{C}(M)$
- $h \overset{(d)}{=} h' \circ \Phi$ if $\Phi : M \to M'$ is an isometry and h and h' are distributed according to $\mathbf{P}_{M,g}$ and $\mathbf{P}_{M',g'}$, resp.

Two dimensions

Let (M, g') be a closed (orientable) surface.

Uniformization Theorem: (M, g') is conformally equivalent to one of the following:

- the sphere;
- the Euclidean plane;
- the hyperbolic plane.

Random surface: Draw a random $\varphi \colon M \to \mathbb{R}$ wrt some prob. measure $\mathbf{P}_{M,g}$

$$g' = e^{\varphi} g$$

Liouville Quantum Gravity:

$$\varphi = \gamma \cdot h, \qquad \gamma > 0, \quad h \text{ is a } \textbf{Gaussian Free Field}$$

Note: h is not a random function.

Gaussian Free Field

Collection of **centered Gaussian random varables**

$$\langle h, \varphi \rangle, \qquad \varphi \in H^1(M)$$

with covariance structure

$$Cov[\langle h, \varphi \rangle, \langle h, \psi \rangle] = \int_M \nabla \varphi \cdot \nabla \psi \, dx$$

Note: Generalization of **Brownian motion** to $2d$.

Crucial: Conformal invariance of the Dirichlet energy

$$\mathfrak{e}_g(u,u) := \int_M |\nabla_g u|^2 \, d\mathrm{vol}_g = \mathfrak{e}_{e^{2\varphi} g}(u,u)$$

Liouville Quantum Gravity

- "Canonical random surfaces"
- Introduced by Polyakov (1980's) in the setting of Bosonic string theory
- Definition $e^{\gamma h} g$ does not make sense, since h is a distribution, not a function.
- Miller, Sheffield '21: Scaling limit of uniform random planar maps if $\gamma = \sqrt{8/3}$ (Brownian map: Le Gall '13, Miermont '13)
- Kahane '85; Duplantier, Sheffield '11; Rhodes, Vargas '14; Shamov '16: Construction of area measure for $\gamma \in (0,2)$ (LQG-measure)

$$\mu_h = e^{\gamma h} \, d\mathrm{vol}$$

- Ding, Dubedat, Dunlap, Falconet '20; Gwynne, Miller '21: Metric analog of $e^{\gamma h} g$, $\gamma \in (0,2) \Rightarrow$ LQG-metric.

Question:

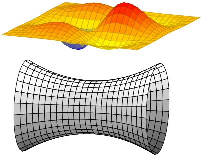
Is there a way to "generalize" this to compact (even)-dimensional manifolds (M, g)?

Random Riemannian Geometry and Conformal Invariance

$\mathbf{P}_g$ is law of Gaussian field, informally given as

$$d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$$

with (non-existing) uniform distribution dh on $\mathcal{C}(M)$, normalizing constant Z_g, and bilinear form $\mathfrak{e}_g(u,v) = (u, Av)_{L^2}$.



Conformal Invariance Requirement

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall \varphi,\ \forall u.$$

In case $n=2$, property of the *Dirichlet energy*

$$\mathfrak{e}_g(u,u) := \int_M \big|\nabla_g u\big|^2 \, d\,\mathrm{vol}_g\,.$$

Random Riemannian Geometry and Conformal Invariance

Gaussian fields $d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(- \frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$ with conformally invariant energy

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall\varphi,\ \forall u.$$

In $n \neq 2$, Dirichlet energy no longer conformally invariant:

$$\mathfrak{e}_{e^{2\varphi}g}(u,u) = \int_M \left|\nabla_g u\right|^2 e^{(n-2)\varphi}\, d\,\mathrm{vol}_g \,.$$

In $n = 4$, more promising: bi-Laplacian energy

$$\tilde{\mathfrak{e}}_g(u,u) := \int_M \left(\Delta_g u\right)^2 d\,\mathrm{vol}_g \,.$$

Still not conformally invariant but close to:

$$\tilde{\mathfrak{e}}_{e^{2\varphi}g}(u,u) := \int_M \left(\Delta_g u + 2\nabla_g\varphi\,\nabla_g u\right)^2 d\,\mathrm{vol}_g = \tilde{\mathfrak{e}}_g(u,u) + \text{ low order terms.}$$

Paneitz '83:

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \Big[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g \cdot |\nabla_g u|^2\Big]\, d\,\mathrm{vol}_g$$

is conformally invariant.

Co-Polyharmonic Energy on n-Manifolds

Assume from now on that (M, g) is n-dimensional smooth, compact, connected Riemannian manifold without boundary, n even.

Graham/Jenne/Mason/Sparling '92:

The co-polyharmonic energy

$$\mathfrak{e}_g(u,v) = c \int_M \big(-\Delta_g\big)^{n/4} u \cdot \big(-\Delta_g\big)^{n/4} v \; d\,\mathrm{vol}_g + \text{low order terms}$$

is conformally invariant.

$\mathfrak{e}_g(u,v) = \int_M p_g u \cdot v \; d\,\mathrm{vol}_g$ with co-polyharmonic operator

$$p_g u := c\,(-\Delta)^{n/2} u + \text{ low order terms}$$

Choose $c = \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}} =: a_n$.

Co-Polyharmonic Energy on n-Manifolds

Integrable functions (or distributions) u on M will be called grounded if $\int_M u\, d\,\mathrm{vol}_g = 0$ (or $\langle u, \mathbf{1}\rangle = \mathbf{0}$, resp.).

Grounded Sobolev spaces $\mathring{H}^s(M,g) = (-\Delta_g)^{-s/2}\mathring{L}^2(M, \mathrm{vol}_g)$ for $s \in \mathbb{R}$,
usual Sobolev spaces $H^s(M,g) = (1-\Delta)^{-s/2}L^2(M, \mathrm{vol}_g) = \mathring{H}^s(M,g) \oplus \mathbb{R}\cdot\mathbf{1}$

Laplacian $-\Delta : H^s \to \mathring{H}^{s-2}$; grounded Green operator $\mathring{\mathsf{G}}_g : \mathring{H}^s \to \mathring{H}^{s+2}$.

Definition

The n-manifold (M,g) is called admissible if $\mathfrak{e}_g > 0$ on $\mathring{H}^{n/2}(M)$.

Large classes of n-manifolds are admissible. For instance in $n = 4$:

- all compact Einstein 4-manifolds with $\mathrm{Ric} \geq 0$ are admissible.
- all compact hyperbolic 4-manifolds with spectral gap $\lambda_1 > 2$ are admissible.

For the sequel, we always assume that (M,g) is admissible.

Key Property of the Co-Polyharmonic Green Kernel

Define co-polyharmonic Green operator

$$\mathsf{k}_g := \mathsf{p}_g^{-1} : H^{-n}(M) \to \mathring{L}^2(M)$$

and associated bilinear form with domain $H^{-n/2}(M)$ by

$$\mathfrak{k}_g(u, v) := \langle u, \mathsf{k}_g v \rangle_{L^2}.$$

Theorem (Dello Schiavo, Herry, K., Sturm '21)

k_g *is an integral operator with an integral kernel* k_g *which is grounded, symmetric, and satisfies*

$$\left| k_g(x, y) + \log d_g(x, y) \right| \leq C_0.$$

Co-Polyharmonic Gaussian Field – Definition, Construction

Definition

A co-polyharmonic Gaussian field on (M, g) is a linear family $\{\langle h, u\rangle\}_{u\in H^{-n/2}}$ of centered Gaussian random variables (defined on some probability space) with

$$\mathbf{E}\big[\langle h, u\rangle \, \langle h, v\rangle\big] = \mathfrak{k}_g(u, v) \qquad \forall u, v \in H^{-n/2}(M).$$

Interpretation: $\mathbf{E}\big[h(x)\big] = 0, \qquad \mathbf{E}\big[h(x)\, h(y)\big] = k_g(x, y) \qquad (\forall x, y)$

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- Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space and $(\xi_j)_{j \in \mathbb{N}}$ an iid sequence of of $\mathcal{N}(0, 1)$ random variables. Furthermore, let $(\psi_j)_{j \in \mathbb{N}_0}$ and $(\nu_j)_{j \in \mathbb{N}_0}$ denote the sequences of eigenfunctions and eigenvalues for p_g (counted with multiplicities).
- The co-polyharmonic field h is given by

$$h := \sum_{j \in \mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

- The co-polyharmonic Gaussian field on (M, g) can be regarded as a random variable with values in $\mathring{H}^{-\epsilon}(M)$ for any $\epsilon > 0$.

Co-Polyharmonic Gaussian Field – Smooth Approximation

Theorem (Dello Schiavo, Herry, K., Sturm '21)

The co-polyharmonic field h is given by

$$h = \sum_{j\in\mathbb{N}} \xi_j \cdot \sqrt{\mathsf{k}_g}\, \psi_j = \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

More precisely,

1. *For each $\ell \in \mathbb{N}$, a centered Gaussian random variable h_ℓ with values in $\mathcal{C}^\infty(M)$ is given by*

$$h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

2. *The convergence $h_\ell \to h$ holds in $L^2(\mathbf{P}) \times H^{-\epsilon}(M)$ for every $\epsilon > 0$. In particular, for a.e. ω and every $\epsilon > 0$,*

$$h^\omega \in H^{-\epsilon}(M),$$

3. *For every $u \in H^{-n/2}(M)$, the family $(\langle u, h_\ell\rangle)_{\ell\in\mathbb{N}}$ is a centered $L^2(\mathbf{P})$-bounded martingale and*

$$\langle u, h_\ell\rangle \to \langle u, h\rangle \quad \text{in } L^2(\mathbf{P}) \text{ as } \ell \to \infty.$$

Co-Polyharmonic Gaussian Field – Smooth Approximation

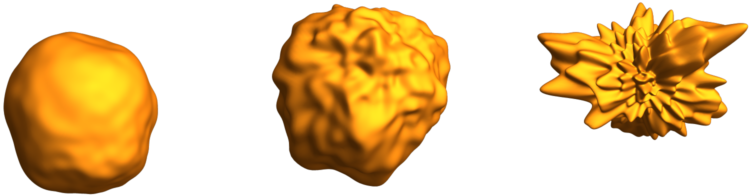

Figure: Courtesy of our co-author Lorenzo Dello Schiavo

Polyharmonic Gaussian Field – Conformal Invariance

The co-polyharmonic Gaussian field is conformally invariant modulo re-grounding.

Theorem (Dello Schiavo, Herry, K., Sturm '21)

Let $h : \Omega \to H^{-\epsilon}(M)$ denote the co-polyharmonic Gaussian field for (M, g) and let $g' = e^{2\varphi} g$ with $\varphi \in \mathcal{C}^\infty(M)$. Then

$$h' := h - \frac{1}{\mathrm{vol}_{g'}(M)} \langle h, \mathbf{1} \rangle_{\mathrm{vol}_{g'}}$$

is the co-polyharmonic Gaussian field for (M, g').

Polyharmonic Gaussian Field – Discrete Approximation

- Let M be the continuous torus $\mathbb{T}^n \cong [0,1)^n$
- Consider its discrete approximations $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$
- Discrete Laplacian

$$\Delta_L f(v) := L^2 \sum_{u \sim v} [f(u) - f(v)]$$

with eigenfunctions $\varphi_z(x) = \frac{1}{\sqrt{2}} \cos(2\pi xz)$ and $\varphi_{-z}(x) = \frac{1}{\sqrt{2}} \sin(2\pi xz)$ and eigenvalues

$$\lambda_{L,z} = 4L^2 \sum_{k=1}^{n} \sin^2\left(\pi z_k / L\right)$$

- Discrete polyharmonic kernel

$$k_L(u,v) = \frac{1}{a_n} \sum_{z \in \mathbb{Z}^n_L \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \cdot \cos\Big(2\pi\, z \cdot (v-u)\Big)$$

where $\mathbb{Z}^n_L = \{z \in \mathbb{Z}^n : \|z\|_\infty < L/2\}$

Polyharmonic Gaussian Field – Discrete Approximation

Polyharmonic Gaussian Field on the discrete torus $\mathbb{T}_L^n$

Centered Gaussian field $(h_L(v))_{v\in\mathbb{T}_L^n}$ with covariance function k_L

Given iid standard normals $(\xi_z)_{z\in\mathbb{Z}_L^n}$, the discrete polyharmonic Gaussian Field is given as

$$h_L = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_{L,z}^{n/4}} \cdot \xi_z\, \varphi_z.$$

The law of the discrete polyharmonic Gaussian Field is given explicitly as

$$c_n \exp\left(-\frac{a_n}{2N}\left\|(-\Delta_L)^{n/4}h\right\|^2\right) d\mathcal{L}^N(h)$$

on $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}_L^n}$ where $N = L^n$.

Polyharmonic Gaussian Field – Discrete Approximation

Theorem (Dello Schiavo, Herry, K., Sturm '23)

- *Convergence of fields $h_L \to h$ as $L \to \infty$: tested against $f \in \bigcup_{s>n/2} H^s(\mathbb{T}^n)$*
- *Convergence of Fourier extension of h_L to h: in each $H^{-\epsilon}(\mathbb{T}^n)$ and also tested against $f \in H^{-n/2}(\mathbb{T}^n)$*

Liouville Geometry

Liouville Geometry

Fix an admissible manifold (M, g) and a co-polyharmonic Gaussian field $h : \Omega \to \mathfrak{D}'$. Our naive goal is to study the 'random geometry' (M, g_h) obtained by the random conformal transformation,

$$g_h = e^{2h} g \,,$$

and in particular to study the associated 'random volume measure' given as

$$\mu^h(x) = e^{nh(x)} \, \mathrm{vol}_g(x)$$

and the 'random metric' (or 'random distance') as

$$d^h(x, y) = \inf_{\varphi} \int_0^1 e^{h(\varphi(t))} |\varphi'(t)| dt$$

Due to the singular nature of the noise h, however, both of these objects will be degenerate – as long as no additional renormalization is built in.

Liouville Geometry

In n=2:
Replacing h by suitable mollifications h_ℓ and proper renormalization leads (for sufficiently small $\gamma \in \mathbb{R}$) to sequences of random measures (μ^{h_ℓ}) and random distances (d^{h_ℓ}) on M which converge as $\ell \to \infty$ to non-trivial limit objects

Duplantier/Sheffield 2011, Rhodes/Vargas 2014

$$\mu^h(x) = \lim_{\ell\to\infty} e^{\gamma h_\ell(x) - \frac{\gamma^2}{2}\mathbf{E}[h_\ell(x)^2]}\,\mathrm{vol}_g(x).$$

Ding/Dubedat/Dunlap/Falconet 2020, Gwynne/Miller 2021

$$d^h(x,y) = \lim_{\ell\to\infty} \frac{1}{\lambda_{\gamma,\ell}} \inf_{\varphi} \int_0^1 e^{\gamma/d_\gamma \cdot h_\ell(\varphi(t))} |\varphi'(t)| dt.$$

Miller/Sheffield 2020/21

For the particular value $\gamma = \sqrt{8/3}$, the random metric measure space (M, d^h, μ^h) is isometric in distribution to the Brownian map
$=$ scaling limit of random triangulations (Le Gall 2013) or quadrangulations (Miermont 2013) of the sphere.

Liouville Quantum Gravity Measure

Let M as before be a compact manifold of even dimension and h the co-polyharmonic Gaussian field.
For $\ell \in \mathbb{N}$ define a random measure $\mu_\ell = \rho_\ell \operatorname{vol}_g$ on M with density

$$\rho_\ell(x) := \exp\left(\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x,x)\right)$$

where as before $h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2} \xi_j \psi_j$ and $k_\ell(x,x) := \mathbf{E}\left[h_\ell^2(x)\right] = \sum_{j=1}^{\ell} \nu_j^{-1} \psi_j^2(x)$.
Based on Kahane 1986, Shamov 2016:

Theorem (Dello Schiavo, Herry, K., Sturm '21)

If $|\gamma| < \sqrt{2n}$, then there exists a random measure μ on M with $\mu_\ell \to \mu$. More precisely, for every $u \in \mathcal{C}(M)$,

$$\int_M u\, d\mu_\ell \longrightarrow \int_M u\, d\mu \quad \text{in } L^1(\mathbf{P}) \text{ and } \mathbf{P}\text{-a.s. as } \ell \to \infty.$$

The random measure $\mu := \lim_{\ell \to \infty} \mu_\ell$ is called *Liouville Quantum Gravity measure*.

Liouville Quantum Gravity Measure

Proposition

If $|\gamma| < \sqrt{n}$, then for every $u \in \mathcal{C}_b(M)$,

$$\left(Y_\ell\right)_{\ell\in\mathbb{N}} := \left(\int_M u\, d\mu_\ell\right)_{\ell\in\mathbb{N}} \quad \text{is } L^2\text{-bounded martingale}$$

Proof: Assume $0 \le u \le 1$. Then

$$\begin{aligned}
\sup_\ell \mathbf{E}\left[{Y_\ell}^2\right] &= \sup_\ell \mathbf{E}\left[\iint e^{\gamma h_\ell(x)+\gamma h_\ell(y)-\frac{\gamma^2}{2}k_\ell(x,x)-\frac{\gamma^2}{2}k_\ell(y,y)} \cdot u(x)u(y)\,dx\,dy\right] \\
&= \sup_\ell \iint e^{\gamma^2 k_\ell(x,y)} \cdot u(x)u(y)\,dx\,dy \\
&\le \iint e^{\gamma^2 k(x,y)}\,dx\,dy \\
&= \iint \frac{1}{d(x,y)^{\gamma^2}}\,dx\,dy + \mathcal{O}(1)
\end{aligned}$$

due to the log divergence of k. The latter integral is finite if and only if $\gamma^2 < n$.

Liouville Quantum Gravity Measure

Theorem (Dello Schiavo, Herry, K., Sturm '21)

If $\gamma < \sqrt{2}$ then a.s. the LQG measure μ does not charge sets of vanishing H^1-capacity

$\longrightarrow$ Dirichlet form $\int_M |\nabla u|^2 d\,\mathrm{vol}_g$ on $L^2(M, \mu)$
$\longrightarrow$ Liouville Brownian motion (random time change of BM)

A key property of the Liouville Quantum Gravity measure is its quasi-invariance under conformal transformations.

Theorem (Dello Schiavo, Herry, K., Sturm '21)

Let μ be the Liouville Quantum Gravity measure for (M, g), and μ' be the Liouville Quantum Gravity measure for (M, g') where $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then

$$\mu' \overset{(\mathrm{d})}{=} e^{(n+\frac{\gamma^2}{2})\varphi}\, \mu.$$

LQG Measure – Discrete Approximation

- Let M be the continuous torus $\mathbb{T}^n \cong [0,1)^n$
- Consider its discrete approximations $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$.
- Recall the polyharmonic Gaussian field on the discrete torus $\mathbb{T}^n_L$

$$h_L = \frac{1}{\sqrt{a_n}} \sum_{z \in \mathbb{Z}^n_L \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/4}} \cdot \xi_z \, \varphi_z.$$

For given $\gamma \in \mathbb{R}$, the **discrete LQG measure** μ_L is the random measure on $\mathbb{T}^n_L$ defined by

$$d\mu_L(v) = \exp\left(\gamma h_L(v) - \frac{\gamma^2}{2} k_L(v,v)\right) dm_L(v),$$

where m_L denotes the normalized counting measure $\frac{1}{L^n} \sum_{u \in \mathbb{T}^n_L} \delta_u$.

In accordance to the approximation of the polyharmonic fields, we have convergence of μ_L to the LQG measure μ on $\mathbb{T}^n$.

LQG Measure – Discrete Approximation

Theorem (Dello Schiavo, Herry, K., Sturm '23)

(i) *For $\gamma < \sqrt{n/e}$ and hierarchically ordered $L = a^\ell$, $a \in \mathbb{N}_{\geq 2}$,*

$$\mu_{a^\ell} \to \mu \qquad \text{in law in } L^1(\mathbf{P}) \text{ as } \ell \to \infty.$$

(ii) *A corresponding result holds true for the LQG measure $\mu_{L,\sharp}$ associated to the Fourier extension $h_{L,\sharp}$: for $\gamma < \sqrt{n/e}$ this semi-discrete LQG measure converges to μ in law in $L^1(\mathbf{P})$ as $L \to \infty$.*

(iii) *An analogous convergence result holds for the so-called spectrally reduced semi-discrete LQG measure in the range $\gamma < \sqrt{2n}$.*

- The range of γ in (i) and (ii) differs from the Gaussian multiplicative chaos construction since this construction uses the eigenvalues of the discrete Laplacian instead of the Laplacian.

Summary

- Construction of co-polyharmonic fields on admissible Riemannian manifolds
- Construction of Liouville Quantum Gravity measures
- Conformal quasi-invariance
- Polyharmonic Gaussian fields and Liouville Quantum Gravity measures on discrete torus
- Scaling limits

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Outlook

- Discrete co-polyharmonic Gaussian fields on manifolds
- Discrete Liouville Quantum Gravity measures on manifolds
- Scaling limits
- Liouville Quantum Gravity metric